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COMMENT

Symmetry in the commensurate anisotropic oscillator

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Abstract. The $SU(m)$ symmetry underlying the degeneracies in the energy levels of the m-dimensional anisotropic oscillator with commensurate frequencies discussed by Rosensteel and Draayer, in the context of models for super-deformed nuclei, is related to the non-bijective canonical transformation found by Moshinsky and his group.

The quantum states of an anisotropic oscillator with commensurate frequencies (with $\omega_i=\omega/\rho_i$ and ρ_i integral and relatively prime), specified by quantum numbers $\{n_i\}$, possessing energies

$$E_{\{n_i\}}=\hbar\omega\sum_i(n_i+\tfrac{1}{2})\frac{1}{\rho_i}$$

are degenerate to the extent that the same energy value can be obtained with more than one integer set $\{n_i\}$. The anisotropic harmonic oscillator has long been of relevance in defining the intrinsic states of rotating deformed nuclei in the Nilsson model [1], but the recently discovered super-deformed high-spin states [2], corresponding to spheroidal nuclear shapes of approximately commensurate axial lengths brings such systems into focus. In particular the symmetry algebra behind the degeneracies has been clarified [3–6], leading to the result that an m-dimensional anisotropic oscillator with commensurate frequencies enjoys an underlying $SU(m)$ symmetry, as for the isotropic case but with the important difference that unlike the latter a given representation occurs not singly but with a multiplicity $\Pi_i\rho_i$. In the present note we comment on this symmetry *vis à vis* a non-bijective canonical transformation.

In a series of papers Moshinsky and his group [7–14] have studied transformations of coordinates (q) and momentum (p) variables, defined implicitly through functional relations

$$H(q,p)=\bar{H}(\bar{q},\bar{p}) \qquad \text{and} \qquad G(q,p)=\bar{G}(\bar{q},\bar{p}) \tag{1}$$

the canonicity of which stand guaranteed provided the necessary and sufficient condition

$$\{H,G\}_{q,p}=\{\bar{H},\bar{G}\}_{\bar{q},\bar{p}} \tag{2}$$

is satisfied with the braces denoting the corresponding Poisson brackets

$$\{H,G\}_{q,p}=\frac{\partial H}{\partial q}\frac{\partial G}{\partial p}-\frac{\partial H}{\partial p}\frac{\partial G}{\partial q}.$$

Consider now the Hamiltonian for the anisotropic oscillator under consideration

$$H=\tfrac{1}{2}\sum_{i=1}^{m}\left(p_i^2+\frac{\omega^2}{\rho_i^2}q_i^2\right)\equiv\sum_{i=1}^{m}h_i \tag{3}$$

where the mass parameter has been put equal to unity. In order to uncover the underlying symmetry a non-bijective implicitly defined canonical transformation will be performed so as to reduce all the frequencies to ω. It is convenient to carry out this operation in two steps: first, a point transformation which is merely a scaling (or dilatation)

$$q_i\to q_i'=\frac{1}{\sqrt{\rho_i}}q_i \qquad \text{with} \qquad p_i\to p_i'=\sqrt{\rho_i}p_i \tag{4}$$

whereupon the Hamiltonian becomes

$$H'=\sum h_i'=\sum_i\frac{1}{2\rho_i}(p_i'^2+\omega^2q_i'^2) \tag{5}$$

and, secondly, by a transformation implemented implicitly through

$$h_i'(q_i',p_i')\equiv\frac{1}{2\rho_i}(p_i'^2+\omega^2q_i'^2)=\tfrac{1}{2}(\bar{p}_i^2+\omega^2\bar{q}_i^2)\equiv\bar{h}_i(\bar{q}_i,\bar{p}_i) \tag{6a}$$

$$g_i'(q_i',p_i')\equiv\rho_i\tan^{-1}\left(\frac{p_i'}{\omega q_i'}\right)=\tan^{-1}\left(\frac{\bar{p}_i}{\omega\bar{q}_i}\right)\equiv\bar{g}_i(\bar{q}_i,\bar{p}_i). \tag{6b}$$

That the transformation is indeed canonical is readily verified by observing that the condition given by equation (2) is satisfied. Thus the Hamiltonian now becomes

$$H(q_i,p_i)=\bar{H}(\bar{q}_i,\bar{p}_i)=\tfrac{1}{2}\sum_i(\bar{p}_i^2+\omega^2\bar{q}_i^2). \tag{7}$$

(It may parenthetically be remarked that the transformation considered would untwist the Lissajous figures (corresponding to classical orbits in space for the two-dimensional case) into ellipses characteristic of superpositions of simple harmonic motions with the same frequency.) The transmutation from anisotropy to isotropy has been achieved at the cost of introducing a foliated (multi-sheeted) structure of the phase space. Thus the $(\omega q_i/\rho_i, p_i)$-plane gets mapped into ρ_i sheets in the $(\omega\bar{q}_i,\bar{p}_i)$ variables. Introducing polar angles θ_i and $\bar{\theta}_i$ to represent corresponding points in the two planes, it may be easily recognized that by virtue of the transformation one has $\bar{\theta}_i=\rho_i\theta_i$ and thus a set of points, ρ_i in number, located at $\theta_i+2\pi s_i/\rho_i$ $(s_i=0,\ldots,\rho_i-1)$ in the original plane get mapped to the same polar angle $\bar{\theta}_i$ (modulo 2π) but lying on ρ_i sheets. These ρ_i points are connected by a group of linear canonical transformations isomorphic to the cyclic group Z_{ρ_i}. Returning to the mapped Hamiltonian, the underlying symmetry is made manifest by observing that the $m(m-1)/2$ angular momenta

$$\bar{L}_{ij}\equiv\bar{q}_i\bar{p}_j-\bar{q}_j\bar{p}_i=\frac{\sqrt{2\bar{h}_i}\sqrt{2\bar{h}_j}}{\omega}\sin(\rho_j\theta_j-\rho_i\theta_i)$$

and the $m(m+1)/2-1$ 'quadrupolar' bilinears

$$\bar{Q}_{ij}\equiv\omega^2\bar{q}_i\bar{q}_j+\bar{p}_i\bar{p}_j=\sqrt{2\bar{h}_i}\sqrt{2\bar{h}_j}\cos(\rho_j\theta_j-\rho_i\theta_i)$$

(discounting $\Sigma_i \bar{Q}_{ii} = 2\,\Sigma_i \bar{h}_i = 2\bar{H}$), constitute all together $m^2 - 1$ constants of motion that generate the symmetry group $SU(m)$ under Poisson bracket operation (their Poisson bracket with $\bar{H}$ vanishes). Furthermore, it is natural to introduce variables

$$A_i = \frac{1}{\sqrt{2\omega}}(\mathrm{i}\bar{p}_i + \omega\bar{q}_i) = \sqrt{\frac{\bar{h}_i}{\omega}}\,\mathrm{e}^{\mathrm{i}\rho_i\theta_i} = \sqrt{\frac{|a_i|^2}{\rho_i}}\left(\frac{a_i}{\sqrt{|a_i|^2}}\right)^{\rho_i} \tag{8}$$

with

$$a_i \equiv \sqrt{\frac{\rho_i}{2\omega}}\left(\mathrm{i}p_i + \frac{\omega}{\rho_i}q_i\right)$$

in terms of which the constants of motion assume the form

$$\bar{L}_{ij} = 2\,\mathrm{Im}(A_i^* A_j) \qquad \text{and} \qquad \bar{Q}_{ij} = 2\omega\,\mathrm{Re}(A_i^* A_j).$$

The foregoing analysis permits a somewhat deeper understanding of the procedure adopted by Rosensteel and Drayer [6] wherein the 'phase operator' is introduced via

$$\hat{\alpha}_i^\dagger = \hat{n}_i^{-1/2}\hat{a}_i^\dagger \qquad \text{and} \qquad \hat{\alpha}_i = \hat{a}_i\hat{n}_i^{-1/2} = (\hat{n}_i + 1)^{-1/2}\hat{a}_i \tag{9}$$

with $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$ being as usual the number operator expressed in terms of the creation annihilation operators, and thence they define

$$\hat{A}_i^\dagger(\rho_i) \equiv \left[\frac{\hat{n}_i}{\rho_i}\right]^{1/2}(\hat{\alpha}^\dagger)^{\rho_i} \tag{10}$$

the parallelism of which with equation (8) is immediately evident. However, here $[\hat{n}_i/\rho_i]$ is the number operator modulo ρ_i whose eigenvalue $[n_i/\rho_i]$ would be the whole integral part of the ratio n_i/ρ_i. This is necessitated by the fact that otherwise $\hat{A}_i(\rho_i)$ and $\hat{A}_i^\dagger(\rho_i)$ would not have satisfied bosonic commutation relations. Moreover analogous to the constants of motion $\bar{L}_{ij}$ and $\bar{Q}_{ij}$ defined above, symmetry operators

$$C_{ij} \equiv \hat{A}_i^\dagger(\rho_i)\hat{A}_j(\rho_j) \tag{11}$$

may be introduced which constitute the generators of a $U(m)$ symmetry which has been shown [6] to be the maximal symmetry algebra for the system.

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